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LOCALIZED RAYLEIGH FUNCTIONS FOR STRUCTURAL AND STRESS ANALYSIS

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Abstract—Localized Rayleigh functions associated with stations introduced throughout a solid or structure allow the established Rayleigh–Ritz energy method to be used in the presence of the cut-outs and discontinuities of practical problems. The resulting method of analysis closely resembles the kinematically-admissible finite element procedure, but the new viewpoint seems to offer some practical as well as conceptual advantages.

As an introduction to the use of these localized functions a simple stress analysis problem is studied. The exact solution of the n degree of freedom Rayleigh problem is obtained and is shown to converge uniformly to the continuum solution as n tends to infinity.

The choice of localized Rayleigh functions for beams and plates and for two-dimensional stress problems is briefly discussed.

1. INTRODUCTION

THE virtues of the Rayleigh–Ritz energy method are well-known, and the method has long been recognized as a powerful yet conceptually simple tool in the analysis of solids and structures. The use of functions running over the totality of a body does however render the method virtually useless in the presence of the cut-outs and discontinuities of practical structures. It would seem that these structures must be analysed numerically on a computer after the introduction of a large number of elements or stations, and it is the purpose of the present paper to introduce the concept of a localized Rayleigh function which allows the established Rayleigh–Ritz procedure to be employed.

The resulting method of analysis retains the recognized virtues of the Rayleigh–Ritz procedure for both static and dynamic problems. Existing and well-tested formulations, linear and nonlinear, can be directly employed. Bound theorems [1] can be expected to hold, and since the Rayleigh amplitudes represent a set of generalized coordinates results of recent general stability theories [2] can be used directly. Use can perhaps also be made of established modifications of the Rayleigh–Ritz procedure, such as that involving the exact solution of the in-plane plate or shell equations.

It has of course been observed by Fraeijs de Veubeke [3] and others that when used with a kinematically admissible displacement field, the finite element method represents an application of the Rayleigh–Ritz procedure, and the exact relationship between the kinematically-admissible finite element method and the proposed method of localized Rayleigh functions is not immediately clear. It seems however that the two methods represent different but heavily overlapping classes of solution, in the sense that most but not all localized Rayleigh analyses could sensibly be described as applications of the finite element technique, and vice-versa. The proposed method of analysis may thus not be entirely new, but it is felt that within the overlap the superior terminology and concepts of the Rayleigh–Ritz procedure should be more fully adopted, while on a more practical plane the localized Rayleigh viewpoint may supply a useful method of fabricating admissible displacement fields for an element.

The proposed use of localized Rayleigh functions is here introduced for a simple stress analysis problem: the exact solution of the n degree of freedom Rayleigh problem is obtained and is shown to converge uniformly to the continuum solution as n tends to infinity. The choice of localized Rayleigh functions for beams and plates and for two-dimensional stress problems is briefly discussed.

2. GENERAL OUTLINE

The proposed method of analysis is, as outlined above, the straight-forward use of the Rayleigh–Ritz energy procedure with localized displacement functions which we shall now describe.

Stations are first introduced throughout the body, and one or more localized Rayleigh functions are associated with each station, these functions having the property that they confer freedom to their own station only. The functions are to be non-zero only in the neighbourhood of their associated station, and some degree of discontinuity in the functions is thus essential.

Two questions immediately arise. How much freedom is to be given to each station, and what degree of discontinuity is to be permitted in the localized Rayleigh functions? The answers to these questions clearly depend on the structural problem under consideration, but even for a given problem the answers are not always clear. We shall thus adopt rather arbitrary answers in the following expositions, simply noting that more freedom and/or less discontinuity can easily be incorporated (albeit at the expense of increased numerical work) if experience proves this to be necessary or desirable.

It is suggested that for ease of computation simple polynomials or circular functions be used for the non-zero regions of the localized Rayleigh functions. Stations can be situated on the boundaries of the body, and the freedom given to these special stations will clearly depend on the boundary conditions of the problem in hand. As with other comparable methods of analysis it is hoped, although certainly not proved, that for a given problem the numerical solutions will converge to the continuum solution as the number of stations is increased, the number necessary to achieve an acceptable solution naturally varying from problem to problem.

We proceed now to introduce further the concept of a localized Rayleigh function by considering a simple one-dimensional stress analysis problem.

3. CONTINUUM FORMULATION

The example chosen to illustrate the proposed method of analysis is that of a uniform elastic bar hanging vertically under its own weight, as shown in Fig. 1. The bar has the constant cross-sectional area A, Young's Modulus E and density ρ. The original length of the bar is L, and a cross-section originally distance x from the support moves down a distance $u(x)$.

The strain energy of the bar can be written as

$$U = \frac{EA}{2}\int_0^L u_x^2 \, dx \tag{1}$$

where a subscript x denotes differentiation with respect to x. The potential energy of the distributed mass of the bar is

$$J = -\rho A \int_0^L u \, dx \tag{2}$$

and we write

$$P \equiv \rho A \tag{3}$$

so that P can be regarded as the magnitude of a generalized force acting through the generalized displacement

$$\mathscr{E} \equiv \int_0^L u \, dx. \tag{4}$$

Setting for convenience $EA = 1$, the total potential energy of the system can be written as

$$\begin{aligned} V &= U - P\mathscr{E} \\ &= \tfrac{1}{2}\int_0^L u_x^2 \, dx - P\int_0^L u \, dx. \end{aligned} \tag{5}$$

The exact solution of the problem, obtained by minimizing V with respect to $u(x)$ with the geometric boundary condition $u(0) = 0$ (or more directly by equilibrium considerations), is readily found to be

$$u = Px(L - \tfrac{1}{2}x) \tag{6}$$

and is shown in Fig. 1.

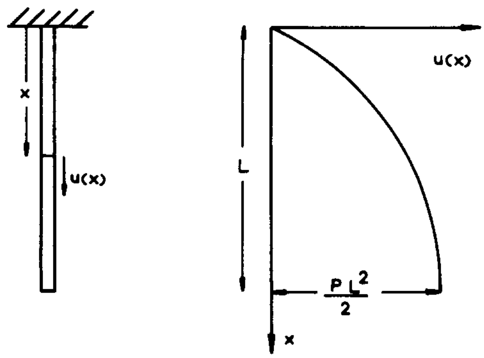

FIG. 1. Continuum solution for the hanging bar.

4. RAYLEIGH–RITZ FORMULATION

To obtain an approximate solution of the problem by the Rayleigh–Ritz procedure it is necessary to write the function $u(x)$ in the form

$$u(x) = \sum_i Q_i f_i(x) \tag{7}$$

the functions $f_i(x)$ satisfying the geometrical boundary conditions of the problem, so that the amplitudes Q_i can be regarded as a set of generalized coordinates.

Normally each $f_i(x)$ would be a continuous function running from $x = 0$ to $x = L$, but it is the essential feature of the proposed method of analysis to choose discontinuous functions each of which is non-zero only in the region of an associated station. We thus introduce n numbered stations along the bar as shown in Fig. 2, making $f_i(x)$, the localized

Rayleigh function associated with a typical station i, non-zero only in the region between the neighbouring stations $i-1$ and $i+1$.

Since the energy functional (5) contains u and u_x but no higher derivatives we must clearly make each localized Rayleigh function continuous in x but can tolerate discontinuities in the first and higher order derivatives. Further, we need give each station only one degree of freedom corresponding to a free choice of the displacement u. The simplest localized Rayleigh functions that we can choose will then be piece-wise linear as shown in Fig. 2.

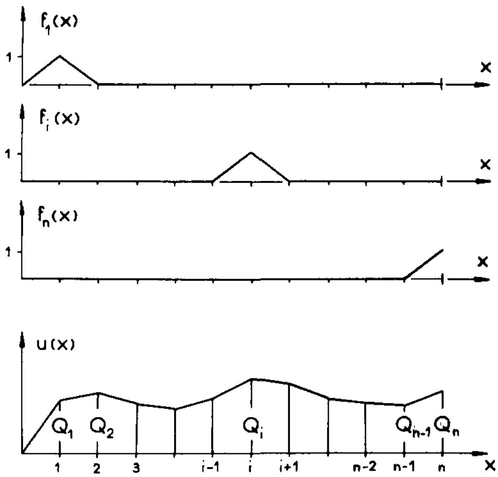

FIG. 2. Localized Rayleigh functions for the bar.

These can be contrasted with the functions which must be used for the normal displacement $w(x)$ of a beam under bending. The energy integral for the beam will contain the second derivative w_{xx}, so that we can no longer tolerate discontinuities in the first derivative w_x. We must then apparently give each station two degrees of freedom corresponding to a free choice of both the displacement w and its first derivative w_x. That is to say we must associate two localized Rayleigh functions $f_i^0(x)$ and $f_i^x(x)$ with a typical station i, these functions having the forms of those shown in Fig. 3. Simple polynomials could be used to

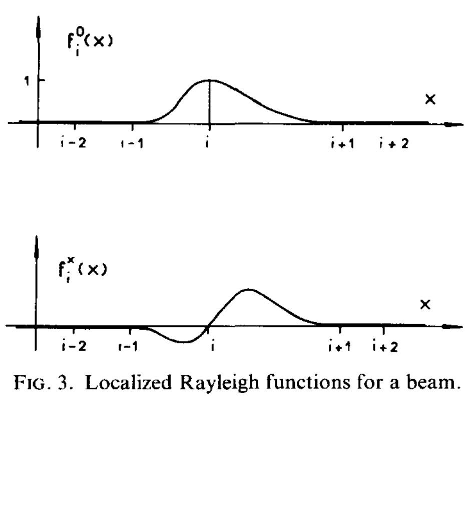

FIG. 3. Localized Rayleigh functions for a beam.

generate these functions, which can be allowed to have discontinuities in their second derivatives at the stations.

Returning to the bar, and having chosen a set of n Rayleigh functions $f_i(x)$, we can now follow the normal Rayleigh–Ritz procedure. Thus substituting $u(x)$ of equation (7) into the energy integral we obtain

$$V(Q_i, P) = U(Q_i) - P\mathscr{E}(Q_i) \tag{8}$$

and since U will be a quadratic and $\mathscr{E}$ a linear function of the Q_i we can write

$$V(Q_i, P) = \tfrac{1}{2}U_{ij}Q_iQ_j - P\mathscr{E}_iQ_i. \tag{9}$$

Here the dummy suffix summation convention is employed with all summations ranging from one to n, and a subscript i denotes differentiation with respect to Q_i. The n equilibrium equations are then

$$V_i(Q_j, P) = U_{ij}Q_j - P\mathscr{E}_i = 0 \tag{10}$$

or in matrix form

$$[U_{ij}][Q_j] = P[\mathscr{E}_i]. \tag{11}$$

5. RAYLEIGH–RITZ SOLUTION

We proceed now to solve these equations for the problem in hand.

For convenience we specify that the stations be equally spaced, so that each local region of the bar has length L/n. Then, identifying a local region by the number of its right-hand station, we see that the potential energy associated with a typical region i is

$$V^i = \tfrac{1}{2}\frac{n}{L}(Q_i^2 - 2Q_iQ_{i-1} + Q_{i-1}^2) - \tfrac{1}{2}P\frac{L}{n}(Q_i + Q_{i-1}) \tag{12}$$

so that

$$V = \sum_{i=1}^{i=n} V^i \tag{13}$$

where Q_0 is to be identified as zero.

Thus the equilibrium equations (11) become

$$\begin{bmatrix} 2 & -1 & & & & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & & & \\ & & -1 & 2 & -1 & & & & & & & & & \\ & & & -1 & 2 & -1 & & & & & & & & \\ & & & & \cdot & \cdot & \cdot & & & & & & & \\ & & & & & \cdot & \cdot & \cdot & & & & & & \\ & & & & & & \cdot & \cdot & \cdot & & & & & \\ & & & & & & & \cdot & \cdot & \cdot & & & & \\ & & & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & & & -1 & 2 & -1 & & \\ & & & & & & & & & & -1 & 2 & -1 & \\ & & & & & & & & & & & -1 & 2 & -1 \\ & & & & & & & & & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Q_{n-4} \\ Q_{n-3} \\ Q_{n-2} \\ Q_{n-1} \\ Q_n \end{bmatrix} = P\frac{L^2}{n^2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} \tag{14}$$

and we see that these equations are satisfied by the values of u predicted to exist at the stations by the exact continuum solution of equation (6). That is to say the equations are satisfied if we write

$$u\left(\frac{iL}{n}\right) = P\frac{iL}{n}\left(L-\frac{iL}{2n}\right) = Q_i. \tag{15}$$

It follows that the Rayleigh–Ritz solutions have the character of inscribed polygons, as shown in Fig. 4, and we see that the solutions will thus converge to the continuum

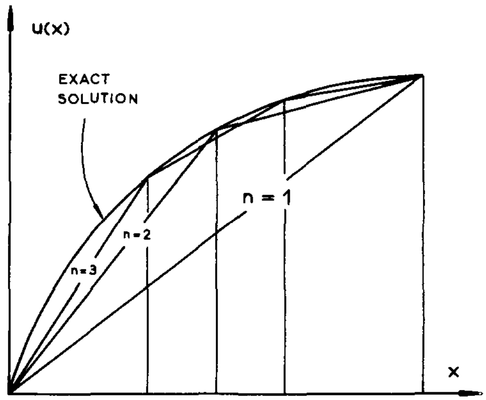

FIG. 4. Rayleigh–Ritz solutions for the bar.

solution as n tends to infinity. Considering the solutions for $n = 1, 2, 4, 8, \ldots$, we see that this convergence satisfies the well-known bound theorem, the value of the generalized deflection $\mathscr{E} \equiv \int_0^L u \, dx$ increasing monotonically with n for a given value of its generalized force P.

6. CONCLUDING REMARKS

We have so far restricted the discussion to localized Rayleigh functions for one displacement function (u for the bar, w for the beam) in one variable (x), but it is felt that the real value of the concept may lie in more complex problems.

In a two-dimensional plane stress problem in the rectangular coordinates (x, y) we might for example draw a rectangular mesh to give us a rectangular array of stations as shown in Fig. 5. We can then introduce two sets of localized Rayleigh functions for the two in-plane displacements $u(x, y)$ and $v(x, y)$. We can tolerate discontinuities in the first derivatives, and for a given displacement (u say) we need perhaps give a station only a single degree of freedom corresponding to a free choice of that displacement. The two sets could then be composed of localized Rayleigh functions with the form of that shown in Fig. 5.

Considering secondly the choice of the normal displacement function $w(x, y)$ for the bending of a plate, we might again introduce a rectangular array of stations as shown in Fig. 6. Clearly we cannot here tolerate discontinuities in the first derivatives, and it seems that we must give each station at least four degrees of freedom corresponding to a free choice

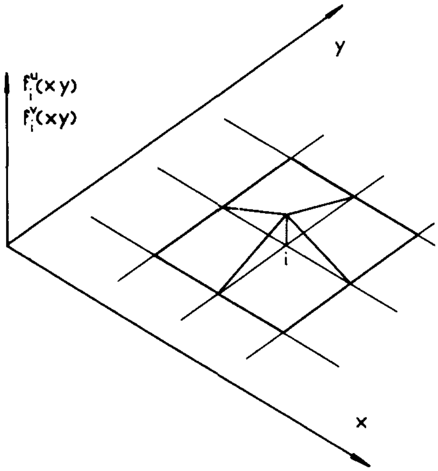

FIG. 5. Localized Rayleigh functions for a plane stress problem.

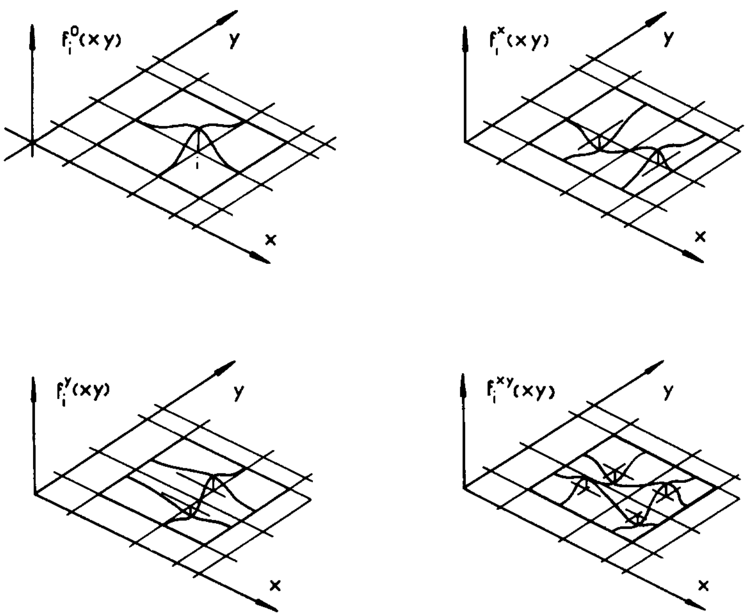

FIG. 6. Localized Rayleigh functions for a plate.

of the displacement w, its first derivatives w_x and w_y, and the cross-derivative w_{xy}, as shown in Fig. 6.

The relationship with the finite element procedure clearly warrants further study, but, whether representing a method of analysis in its own right, or more simply a re-interpretation of the kinematically-admissible finite element procedure, it is felt that the concept of a localized Rayleigh function can play a useful role in the analysis of practical solids and structures.

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Résumé—Des fonctions localisées de Rayleigh associées à des stations introduites à travers un solide ou une charpente permet à la méthode d'énergie Rayleigh–Ritz établie d'être employée en présence des découpures et des discontinuités des problèmes pratiques. La méthode d'analyse y résultant peut être employée avec une machine à calculer et semble avoir un grand nombre d'advantages sur des méthodes comparables.

Comme introduction à l'emploi de ces fonctions localisées un simple problème d'analyse de tension est étudié. La solution exacte du degré n de liberté du problème Rayleigh est obtenue et montrée convergeant uniformément vers la solution continuum alors que n se tend vers l'infini.

Le choix de fonctions Rayleigh localisées pour des poutres et des plaques pour des problèmes de tension à deux dimensions est brèvement discuté.

Zusammenfassung—Lokalisierte Rayleigh Funktionen in Verbindung mit Standorten die in Festkörper oder Strukturen eingeführt werden ermöglichen die Anwendung der üblichen Rayleigh–Ritz Energiemethode auch beim Vorhandensein von Ausschnitten und Diskontinuitäten—in praktischen Problemen. Die gewonnene Analysen-Methode eignet sich für Digitalrechner und hat anscheinend mehrere Vorteile im Vergleich mit ähnlichen Methoden.

Als Einführung zu dieser Methode wird ein einfaches Problem der Spannungsanalyse behandelt. Die genaue Lösung des Rayleigh Problemes mit Freiheit n-ten Grades wird gegeben, ferner wird gezeigt, dass diese Lösung dem Kontinuum annähert wenn n zur Unendlichkeit neigt.

Die Auswahl lokalisierter Rayleigh Funktionen für Balken und Platten sowie für zweidimensionale Spannungsprobleme wird kurz behandelt.

Абстракт—Локализированные функции (Rayleigh) Рейлега, соответствующие пунктам, находящимся повсюду в твёрдом теле или в структуре, позволяют применять установленный метод энергии (Rayleigh-Ritz) Рейлега—Ритца при наличии выключений и нарушений непрерывности практических проблем. Полученный в результате метод анализа удовлетворяет требованиям для применения с цифровой вычислительной машиной (компютором) и, кажется, представляет некоторые преимущества перед сравнимыми методами.

Как введение к применению этих локализированных функций, изучается проблема анализа простого напряжения. Получено точное решение n степени свободы проблемы (Rayleigh) Рейлега и показано, как свести единообразно к постоянному решению в то время, когда n стремится к бесконечности.

Кратко обсуждается выбор локализированных функций (Rayleigh) Рейлега для балок и пластин и для проблем двумерного напряжения.